Perturbed Spherically Symmetric Dust Solution of the Field Equations in Observational Coordinates with Cosmological Data Functions

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Abstract

Using the framework for solving the spherically symmetric field equations in observational coordinates given in Araújo & Stoeger (1999), their formulation and solution in the perturbed FLRW spherically symmetric case with observational data representing galaxy redshifts, number counts and observer area distances, both as functions of redshift on our past light cone, are presented. The importance of the central conditions, those which must hold on our world line $\mathcal{C}$, is emphasized. In detailing the solution for these perturbations, we discuss the gauge problem and its resolution in this context, as well as how errors and gaps in the data are propagated together with the

genuine perturbations. This will provide guidance for solving, and interpreting the solutions of the more complicated general perturbation problem with observational data on our past light cone.

1 Introduction

In two recent papers ([Araújo and Stoeger (1999), Araújo et al. (2001)]) we demonstrated in detail how to solve exactly the Einstein field equations for dust in observational coordinates with cosmological data function representing galaxy redshifts, and observer area distances and galaxy number counts as functions of redshift. These data are given, not on a space-like surface of constant time, but rather on our past light cone $C^{-}(p_0)$, which is centered at our observational position p_0 "here and now" on our world line $\mathcal{C}$. These results demonstrate how cosmologically relevant astronomical data can be used to determine the space-time structure of the universe – the cosmological model which best fits our universe. This has been the aim of a series of papers going back to the Physics Reports paper by Ellis *et al.*, (1985) The motivation and history of this "observational cosmology program" is summarized in Araújo & Stoeger (1999).

The primary aim of this program is to strengthen the connections between astronomical observations and cosmological theory. We do this by allowing observational data to determine the geometry of spacetime as much as possible, *without* relying on *a priori* assumptions more than is necessary or justified. Basically, we want to find out not only how far our observable universe is from being isotropic and spatially homogeneous (that, is describable by a Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological model) on various length scales, but also to give a dynamic account of those deviations ([Stoeger et al. (1992a)]).

Although there are strong indications, especially from the character of the cosmic microwave background radiation (CMWBR), ([Stoeger et al. (1995), Stoeger et al. (1997)]) that our universe is very close to being FLRW on the largest length scales, it is very clear that, since it is lumpy on small and intermediate length scales, it is not exactly FLRW. In light of this it is very important to determine, from the data we in principle have available, how the universe deviates from FLRW on the largest scales, and how those deviations grow or damp as we move off our past light cone into the past, or into the future, as well as the degree to which the errors and gaps in the

data induce imprecisions in the cosmological model we adopt. Genuine perturbations from FLRW and errors in the data will be propagated together ([Stoeger et al. (1992a)]). In practice, they will have to be separated out on $C^{-}(p_0)$ and then followed individually in time. Both of these goals are pursued in this paper and in a subsequent paper, where we show in detail how to solve the perturbed field equations with cosmological data on $C^{-}(p_0)$ with a best-fit FLRW model as the background, or zeroth- order, solution. In this paper we limit ourselves to spherically symmetric perturbations, using the framework we developed in our recent exact spherically symmetric treatment ([Araújo and Stoeger (1999)]). In the subsequent paper we shall tackle the case of general perturbations. These two cases were previously studied in two much earlier papers, ([Stoeger et al. (1992a), Stoeger et al. (1992b)]) but unsuccessfully, due to an important error which led to the severe over-restriction of the solutions (see Araújo & Stoeger (1999) and references therein).

What is the relevance of constructing spherically symmetric perturbed solutions to FLRW, when we could use the same data to find the *exact* spherically symmetric solution ([Araújo and Stoeger (1999)])? It is obviously very important to understand observationally based perturbations to FLRW in their own right. Having a secure approach for determining exact spherically symmetric solutions enables us to study and interpret spherically symmetric perturbations very simply and carefully, including the gauge problem (see Section 4) and the propagation of errors and gaps in the data, as mentioned above, before going on to solve and and interpret the more complicated general perturbation case in this observational context. Finally, studying observationally determined spherically symmetric perturbations will help us define criteria for the use of perturbation theory itself – or, equivalently, for defining a class of 'almost FLRW' models and for determining whether or not our observable universe is 'almost FLRW,' and on what length scale, in a meaningful and rigorous way. Certainly, it is not 'almost FLRW' on small and many intermediate length scales, but there is strong evidence from CMWBR measurements that it is on the largest scales, as we indicated above. At what length scale can we begin to use 'almost FLRW' models to describe the universe? Using this approach, we should be able eventually to answer this important question. Further discussion of the philosophy of this approach is given in Stoeger *et al.*, (1992a).

In Section 2, we briefly summarize our characterization of observational coordinates, as well as the metric and observational relations in observational

coordinates, within the framework of spherically symmetric deviations from FLRW data. In Section 3, we give the exact spherically symmetric field equations in observational coordinates, and in section 4, we discuss gauge problem as it relates to our observationally based cosmological perturbations from FLRW. We describe in Section 5 the first step of the integration procedure, which is the solution of the perturbed null Raychaudhuri equation to determine the redshift z as a function of the null radial coordinate y, that is to find $z = z(y)$. Finally, in Section 6, we complete the integration procedure by explicitly solving for all the spherically symmetric perturbations in terms of the data functions of $z_1(y)$, the non-FLRW component of $z(y)$, and we briefly discuss the application of these results to the issues we have just highlighted.

2 Coordinates, Metric and Observational Relations

We are using observational coordinates (which were first suggested by Temple (1938)). As described by Ellis *et al.*, (1985) these coordinates $x^i = \{w, y, \theta, \phi\}$ are centered on the observer's world line $\mathcal{C}$ and defined in the following way:
(i) w is constant on each past light cone along $\mathcal{C}$, with $u^a\partial_a w > 0$ along $\mathcal{C}$, where u^a is the 4-velocity of matter ($u^a u_a = -1$). In other words, each $w = constant$ specifies a past light cone along $\mathcal{C}$. Our past light cone is designated as $w = w_0$.
(ii) y is the null radial coordinate. It measures distance down the null geodesics – with affine parameter ν – generating each past light cone centered on $\mathcal{C}$. $y = 0$ on $\mathcal{C}$ and $dy/d\nu > 0$ on each null cone – so that y increases as one moves down a past light cone away from $\mathcal{C}$.
(iii) θ and ϕ are the latitude and longitude of observation, respectively – spherical coordinates based on a parallelly propagated orthonormal tetrad along $\mathcal{C}$, and defined away from $\mathcal{C}$ by $k^a\partial_a\theta = k^a\partial_a\phi = 0$, where k^a is the past-directed wave vector of photons ($k^a k_a = 0$).
There are certain freedoms in the specification of these observational coordinates. In w there is the remaining freedom to specify w along our world line $\mathcal{C}$. Once specified there it is fixed for all other world lines. There is considerable freedom in the choice of y – there are a large variety of possible choices for this coordinate – the affine parameter, the redshift z, the observer

area distance $C(w,y)$ itself, which we oftentimes write as $r_0(z)$ or $r_0(y)$ on our past light cone $w = w_0$. We normally choose y to be comoving with the fluid, that is $u^a\partial_a y = 0$. Once we have made this choice, there is still a little bit of freedom left in y, which we shall use below. The freedom in the θ and ϕ coordinates corresponds to a rigid rotation of orthonormal tetrad at one point, say p_0, on our world line $\mathcal{C}$. They are just spherical coordinates on the celestial sphere with respect to the (physically non-rotating) reference frame of the orthonormal tetrad.

Since we are using the best-fit FLRW model as our background spacetime the most general pertubed metric in observational coordinates takes the form:

$$g_{\mu\nu} = \begin{pmatrix} -R^2+Z^2 & R^2+\beta^2 & v_2 & v_3 \\ R^2+\beta^2 & 0 & 0 & 0 \\ v_2 & 0 & R^2\hat{f}^2+h_{22} & h_{23} \\ v_3 & 0 & h_{23} & R^2\hat{f}^2\sin^2\theta+h_{33} \end{pmatrix} \tag{1}$$

where $R^2, R^2\hat{f}^2$ and $R^2\hat{f}^2\sin^2\theta$ are the FLRW values of the respective metric components, taken here as zeroth-order terms, and all the other terms are the non-FLRW perturbations in the sense described in the introduction. $\hat{f}$ is given by

$$\hat{f}(y) = \begin{cases} \sin y & k=1 \quad \text{(closed)} \\ y & k=0 \quad \text{(flat)} \\ \sinh y & k=-1 \quad \text{(open).} \end{cases} \tag{2}$$

In this paper we are concerned with spherical perturbations. The metric then is a lot simplified with

$$v_2 = v_3 = 0 \tag{3}$$

$$h_{23} = 0 \tag{4}$$

$$h_{22} = h_{33}(\sin\theta)^{-2}. \tag{5}$$

Therefore, the spherically symmetric perturbation problem is reduced to solving the linearized field equations for $\beta^2(w,y)$, $Z^2(w,y)$ and $h_{22}(w,y)$.

The remaining coordinate freedom which preserves the observational form of the metric is a scaling of w and of y:

$$w \to \tilde{w} = \tilde{w}(w) \ , \quad y \to \tilde{y} = \tilde{y}(y) \qquad \left(\frac{d\tilde{w}}{dw} \neq 0 \neq \frac{d\tilde{y}}{dy} \right). \tag{6}$$

The first, as we mentioned above, corresponds to a freedom to choose w as any time parameter we wish along $\mathcal{C}$, along our world line at $y = 0$. This is effected in this case by choosing $(-g_{00}(w, 0))^{1/2}$. The second corresponds to the freedom to choose y as any null distance parameter on an initial light cone – typically our light cone at $w = w_0$. Then that choice is effectively dragged onto other light cones by the fluid flow – y is comoving with the fluid 4-velocity, as we have already indicated. We shall use this freedom to choose y by setting:

$$\beta^2(w_0, y) = -Z^2(w_0, y) \tag{7}$$

We note that the first attempt to solve this problem by [Stoeger et al. (1992a)] is in error as it relies on the assumption that $\beta^2(w, y) = -Z^2(w, y)$, which is too restrictive when y is comoving.

Before we proceed we write down our observational quantities in terms of the coordinate metric components given above:

(i) Redshift. The redshift z at time w_0 on $\mathcal{C}$ for a comoving source a null radial distance y down $C^-(p_0)$ is given by

$$1 + z = A_0[R^2(w_0, y) - Z^2(w_0, y)]^{-1/2}, \tag{8}$$

where, $A_0 \equiv [R^2(w_0, 0) - Z^2(w_0, 0)]^{1/2}$. Expanding (8) to first order using the binomial theorem yields

$$\frac{A_0}{1 + z} = R(w_0, y) - \frac{1}{2}\frac{Z^2(w_0, y)}{R(w_0, y)} \tag{9}$$

This is just the observed redshift, which is directly determined by source spectra, once they are corrected for the Doppler shift due to local motions.

(ii) Observer Area Distance. The observer area distance, often written as r_0, measured at time w_0 on $\mathcal{C}$ for a source at a null radial distance y is given by the expression ${r_0}^4 sin^2\theta = \det(g_{IJ})$, I,J ranging over the values 2,3 ([Ellis et al. (1985)]). In our case, it takes the form:

$$r_0 = [R^2(w_0,y)\hat{f}^2 + h_{22}(w_0,y)]^{1/2} = R(w_0,y)\hat{f}(y) + \frac{h_{22}}{2R(w_0,y)\hat{f}(y)} + \dots \quad (10)$$

(iii) Galaxy Number Counts. The number of galaxies counted by a central observer out to a null radial distance y is given by (see [Ellis (1971), Ellis et al. (1985)]):

$$N(y) = 4\pi \int_0^y \mu(w_0,\tilde{y}) m^{-1} \left(R_0^2(\tilde{y}) - Z_0^2(\tilde{y})\right)^{1/2} r_0(\tilde{y})^2 d\tilde{y} \quad (11)$$

where μ is the mass-energy density, m is the average galaxy mass, $R_0 \equiv R(w_0,y)$ and $Z_0^2 \equiv Z^2(w_0,y)$. Then the total energy density can be written as

$$\mu(w_0,y) = m\, n(w_0,y) = M_0(z)\, \frac{dz}{dy}\, \frac{1}{(R_0^2 - Z_0^2)^{1/2}} \quad (12)$$

where $n(w_0,y)$ is the number density of sources at (w_0,y), and where

$$M_0(z) \equiv \frac{m}{J}\, \frac{1}{d\Omega}\, \frac{1}{r_0^2}\, \frac{dN}{dz}. \quad (13)$$

Here $d\Omega$ is the solid angle over which sources are counted, and J is the completeness of the galaxy count, that is, the fraction of sources in the volume that are counted is J. The effects of dark matter in biasing the galactic distribution may be incorporated via J. In particular, strong biasing is needed if the number counts have a fractal behaviour on local scales ([Humphreys et al. (1998)]).

3 The spherically symmetric field equations in observational coordinates

In this section we present the exact spherically symmetric field equations ([Araújo and Stoeger (1999)]) in observational coordinates for the case of dust. The perturbed spherically symmetric field equations can be constructed from them, as we show in the sequel.

In observational coordinates the spherically symmetric metric takes the general form:

$$ds^2 = -A(w,y)^2 dw^2 + 2A(w,y)B(w,y)dwdy + C(w,y)^2 d\Omega^2. \quad (14)$$

The central conditions for the metric variables $A(w,y)$, $B(w,y)$ and $C(w,y)$ in (14) – that is, their proper behavior as they approach $y=0$ are:

$$\text{as } y\to 0: \qquad \begin{aligned} &A(w,y)\to A(w,0)\neq 0,\\ &B(w,y)\to B(w,0)\neq 0,\\ &C(w,y)\to B(w,0)y=0,\\ &C_y(w,y)\to B(w,0). \end{aligned} \tag{15}$$

As pointed out in Araújo & Stoeger (1999), in this exact case the freedom to choose the coordinate y is used by setting

$$A(w_0,y)=B(w_0,y) \tag{16}$$

Also the redshift at time w_0 is given by

$$1+z=\frac{A_0}{A(w_0,y)} \tag{17}$$

In our present notation A, B and C are identified as

$$A \equiv [R^2(w,y)-Z^2(w,y)]^{1/2} \tag{18}$$
$$B \equiv \frac{[R^2(w,y)+\beta^2(w,y)]}{[R^2(w,y)-Z^2(w,y)]^{1/2}} \tag{19}$$
$$C \equiv [R^2(w,y)\hat{f}^2(y)+h_{22}(w,y)]^{1/2}. \tag{20}$$

As firstly pointed out by Stoeger *et al.*, (1992c) the general spherically symmetric field equations for dust in terms of the metric (14) split in two sets of radial and time equations.

The time independent equations are

$$\frac{\ddot{C}}{C} = \frac{\dot{C}}{C}\frac{\dot{A}}{A}+\omega A^2 \tag{21}$$
$$\frac{\ddot{B}}{B} = \frac{\dot{B}}{B}\frac{\dot{A}}{A}-2\omega A^2-\frac{1}{2}\mu A^2, \tag{22}$$

where an overdot indicates $\partial/\partial w$ and the quantities ω and μ above are given by

$$\mu(w,y) = \mu_0(y)\; B^{-1}(w,y)\; C^{-2}(w,y) \tag{23}$$

$$\omega(w,y) = \frac{\omega_0(y)}{C^3(w,y)} = -\frac{1}{2C^2} + \frac{\dot{C}}{AC}\frac{C'}{BC} + \frac{1}{2}\left(\frac{C'}{BC}\right)^2, \tag{24}$$

where a prime indicates $\partial/\partial y$, μ again is the relativistic mass-energy density of the dust and ω_0 is a quantity closely related to μ_0 (see equation (26) below). Both ω_0 and μ_0 are specified by data on our past light cone.

The radial equations are

$$\frac{C''}{C} = \frac{C'}{C}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{2}B^2\mu \tag{25}$$

$$\omega_0'(y) = -\frac{1}{2}\mu_0(y)\left(\frac{\dot{C}}{A} + \frac{C'}{B}\right) \tag{26}$$

$$\frac{(\dot{C})'}{C} = \frac{\dot{B}}{B}\frac{C'}{C} - \omega AB \tag{27}$$

The contracted Bianchi identities yields

$$A' + \dot{B} = 0. \tag{28}$$

From 26 we see that there is a naturally defined "potential"

$$W(y) \equiv \frac{\dot{C}}{A} + \frac{C'}{B}, \tag{29}$$

that simplifies the integration procedure since it implies that the time-derivative equations (21 and 22) are not necessary to obtain the solution becoming consistency conditions ([Araújo and Stoeger (1999)]).

Stoeger *et al.*, (1992c) and Maartens *et al.*, (1996) have shown that equations (24) and (29) can be transformed into equations for A and B, thus reducing the problem to determining C:

$$A = \frac{\dot{C}}{[W^2 - 1 - 2\omega_0/C]^{1/2}} \tag{30}$$

$$B = \frac{C'}{W - [W^2 - 1 - 2\omega_0/C]^{1/2}}. \tag{31}$$

Now, as pointed out by Araújo and Stoeger (1999), if we temporarily choose $y = z$, which is always legitimate, knowing $C(w_0, z)$ from the data, $\omega_0(z)$ can be obtained from equation (31) and has the form

$$\omega_0 = \frac{CW^2}{2}\left\{1 - \frac{1}{W^2} - \left[1 - \frac{(1+z)C'}{WA_0}\right]^2\right\} \tag{32}$$

with $y = z$, and we have used (16) and (17) to write

$$B(w_0, y) = A(w_0, 0)/(1+z). \tag{33}$$

4 The Gauge Problem

Before we proceed to integrate the perturbed spherically symmetric field equations we address in this section the gauge problem, a problem that arises when one considers perturbed models in General Relativity due to the fact that there is no invariant way to define a background space-time given the actual space-time.

Perturbations are usually discussed by assuming that the space-time (M, g) admits a family of coordinate systems in which the metric tensor components can be written as

$$g_{\mu\nu}(x^\sigma, \eta) = \overset{\circ}{g}_{\mu\nu}(x^\sigma) + \eta h_{\mu\nu}(x^\sigma) + \frac{1}{2!}\eta^2 k_{\mu\nu}(x^\sigma) + \frac{1}{3!}\eta^3 l_{\mu\nu}(x^\sigma) + \ldots \tag{34}$$

where $g, h, k, l \ldots$, are of the same magnitude and $\eta \ll 1$. Here we are comparing a point in the space-time (M, g) with coordinates x^σ to a point in another space-time (background) $(M_0, \overset{\circ}{g})$ with the same coordinates.

Following Stewart and Walker (1974), we adopt in the present discussion the point of view that the perturbed space-time (M, g) is to be thought of as the result of some slight changes made in the background space-time $(M_0, \overset{\circ}{g})$. This approach emphasizes the notion that the background and the perturbed space-times are considered as distinct objects. One also requires the continuity of the perturbation by assuming that $(M_0, \overset{\circ}{g})$ and (M, g) are connected by a path in the space of space-times. Therefore, since the perturbation is small, we consider sequences of diffeomorphic space-times depending on the

continuously varying parameter η. As we shall see below, the concepts of coordinate and gauge transformations, as well as the linearization of the field equations, are made more transparent when viewed in this light.

In order to formalise these ideas ([Geroch (1969)]), we assume there exists a smooth, Hausdorff, five dimensional manifold M in which the space-times under consideration are smooth, nonintersecting, properly-embedded four-dimensional submanifolds. In other words, we are considering a one-parameter family of space-times $[M_\eta, g(\eta)]$ embedded in five-dimensional manifold M. The parametrisation is chosen in such a way that the background space-time is given by $\eta = 0$.

We now consider any smooth, nowhere-vanishing vector field ν on M which is transversal (nowhere tangent) to the M_η. This vector field naturally defines a map, $\Phi_\eta : M_0 \to M_\eta$, between the background and the perturbed space-times which identifies points lying on the same integral curve of ν. This map is called the identification map because it actually defines when points in different space-times are to be regarded as the same. A choice of vector field ν is called a choice of identification gauge.

Having made these remarks, we now recognise (34) as arising from a choice of identification gauge such that $\Phi_\eta : M_0 \to M_\eta$ maps points having the same coordinates.

We consider two kinds of coordinate transformations within the family of coordinate systems in which (34) holds. Under the first we make the same coordinate changes on both M_0 and M_η and keep identifying points with the same coordinates, i.e. all terms in (34) transform as tensors. Of course, $h_{\mu\nu}, k_{\mu\nu}, \dots$ are not tensors by themselves but rather parts of $g_{\mu\nu}$. However, in analogy to the weak field approximation in which the background space-time is flat, the idea is to regard $h_{\mu\nu}, k_{\mu\nu}, \dots$ as tensor fields on the background space-time. This transformation is therefore, the strong field analogue to the background Lorentz transformations of the weak field approximation. Clearly, it does not affect the ordering scheme of (34) and it is obvious that we are still in the same identification gauge. So, the coordinate freedoms mentioned in section 2 fall into this class.

Under the second kind of coordinate transformations we make different coordinate changes on M_0 and M_η and still identify points with the same coordinates. To illustrate this important point, suppose we make a small coordinate transformation on M_0 of the form

$$x^\mu \to x^{'\mu} = x^\mu + \eta \xi^\mu \tag{35}$$

where ξ^μ is an arbitrary vector field on M_0. On M_{eta} we do not make any coordinate transformation. Then, neglecting terms of $O(\eta^2)$, we find ([Landau and Lifshits (1975)])

$$h_{\mu\nu} - h'_{\mu\nu} = 2\xi_{(\mu;\nu)} = (\pounds_\xi \overset{\circ}{g})_{\mu\nu} \tag{36}$$

where $\pounds_\xi$ is the Lie derivative with respect to the vector field ξ. We observe from this example that the whole effect of the coordinate transformation is expressed in $h_{\mu\nu}$. This coordinate transformation also does not alter the ordering scheme given by (34), so we are still in the assumed family of coordinate systems. However, it shows that $h_{\mu\nu}$ is not fixed uniquely, or equivalently, that the background space-time is only fixed up to small transformations of this kind. These transformations are called gauge transformations. Under a gauge transformation all quantities dependent upon $h_{\mu\nu}$ in general also undergo transformations. It is clear from (36) that if a quantity vanishes in the background space-time, then it is gauge invariant to all orders.

In our case here we are using a best-fit FLRW model as our background space-time, and, as we have pointed out above, the further coordinate transformations allowed, respecting this choice, are those we have labelled as type 1. As we shall see below (Section 6), given the data, and the coordinate choices we have insisted upon, along with the central conditions, equations (15), there is practically no coordinate freedom left. Essentially, our observational frame of reference and the coordinates we use to express that along with the choice of the FLRW background by best-fitting methods is equivalent to choosing the gauge. From this point of view, then, we are not really free to choose any gauge we wish. The choice is imposed by our observational situation and by the data we use in determining the best-fit FLRW model. Thus, though it may be may still be important to identify and use gauge-invariant quantities to resolve some issues, the perturbed quantities specified by the observational situation are observationally based and observable, and thus will reflect both the deviations of our space-time from FLRW and the errors and lacunae in the data used to determine our best-fit FLRW model. With regard to these latter, we are automatically led to modelling these as the imprecision or uncertainty in our identification of the background FLRW space-time using the data we have available.

5 Perturbed null Raychaudhuri equation

Our first step in solving the perturbed spherically symmetric field equations is the solution of the perturbed version of the null Raychaudhuri equation (25) on our light cone $w = w_0$ to find the redshift $z = z(y)$. All of our observational data on $C^-(p_0)$ is given as a function of z. In order to proceed with the integration we need to find $z = z(y)$, so that we can give the data as a function of y for $w = w_0$. Following Stoeger *et al.*, (1992c), since $C(w_0, y) = r_0$, the exact null Raychaudhuri equation (25) can be written as

$$\frac{r_0''}{r_0} = 2\frac{r_0'}{r_0}\frac{A'}{A} - \frac{1}{2}A^2\mu_0, \tag{37}$$

and after some manipulation put in the form:

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[z'\frac{\mathrm{d}r_0}{\mathrm{d}z}(1+z)^2\right] = -\frac{1}{2}A_0 r_0(1+z)M_0(z)\frac{\mathrm{d}z}{\mathrm{d}y} \tag{38}$$

This equation has a first integral:

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}y} &= A_0\left(\frac{\mathrm{d}r_{0_F}}{\mathrm{d}z} + \frac{\mathrm{d}r_{0_+}}{\mathrm{d}z}\right)^{-1}(1+z)^{-2} \\ &\quad \times \left\{1 - \frac{1}{2}\int_0^z (1+z)(r_{0_F} + r_{0_+})(M_{0_F} + M_{0_+})\mathrm{d}z\right\}, \end{aligned}\tag{39}$$

where we have written the data for the spherically symmetric perturbation problem in the form

$$r_0(z) = r_{0_F}(z) + r_{0_+}(z) \tag{40}$$

$$M_0(z) = M_{0_F}(z) + M_{0_+}(z). \tag{41}$$

Here the subscripts F and $+$ denote the FLRW and non-FLRW components of the data respectively. It is important to point out ([Stoeger (1987), Ellis and Stoeger (1987), Stoeger et al. (1992c)] that in the case of FLRW both r_{0_F} and M_{0_F} have very particular functional forms and if observations cannot be fit by those functional forms the universe is not FLRW.

It was shown by Stoeger *et al.*, (1992a) that considering r_{0_F} and M_{0_F} as zeroth-order quantities and r_{0_+} and M_{0_+} as first-order quantities, we can

write equations for the successive orders in the perturbation of dz/dy as follows. Zeroth-order equation:

$$\left(\frac{\mathrm{d}z}{\mathrm{d}y}\right)_F = A_0 \left(\frac{\mathrm{d}r_{0_F}}{\mathrm{d}z}\right)^{-1} (1+z)^{-2} \times \left\{1 - \frac{1}{2}\int_0^z (1+z) r_{0_F} M_{0_F}(z) \mathrm{d}z\right\} \quad (42)$$

First order equation:

$$\begin{aligned}\left(\frac{\mathrm{d}z}{\mathrm{d}y}\right)_1 \quad &= \quad -A_0 \left[\frac{\mathrm{d}r_{0_+}/\mathrm{d}z}{(\mathrm{d}r_{0_F}/\mathrm{d}z)^2}\right] (1+z)^{-2} \left\{1 - \frac{1}{2}\int_0^z (1+z) r_{0_F}(z) M_{0_F}(z) \mathrm{d}z\right\} \\ &+ A_0 \left(\frac{\mathrm{d}r_{0_F}}{\mathrm{d}z}\right)^{-1} (1+z)^{-2} \\ &\times \left\{-\frac{1}{2}\int_0^z (1+z)(r_{0_F}(z) M_{0_+}(z) + r_{0_+}(z) M_{0_F}(z)) \mathrm{d}z\right\}. \quad (43)\end{aligned}$$

Integrating equations (42) and (43) yields

$$z_F = z_F(y) \qquad z_1 = z_1(y), \quad (44)$$

and to first-order we write

$$z(y) = z_F(y) + z_1(y). \quad (45)$$

Obviously, in the same way we can go to higher orders in the perturbation series. In this paper, we restrict ourselves to first order. Now that we have shown how to determine the redshift z as a function of y from the data functions, we can proceed to find the perturbed metric functions in terms of y and w.

Before doing so, however, it is helpful to recognize that the perturbed data functions $r_{0_+}(z)$ and $M_{0_+}(z)$, as well as their FLRW background counterparts, will themselves be constructed from discrete data – with their many gaps and errors – by fitting some continuous function to them, a power law, or a polynomial, for instance ([Stoeger et al. (1992a)]). More precise or more complete data will necessitate a new fitting, to obtain improved data functions. The amount of uncertainty in the solutions due to gaps and errors

in the data will then be represented by a component of the perturbed metric functions. These can be tracked separately from the genuine deviations from FLRW simply by labelling the errors in the data functions by the usual indicators (e. g. error bars) and then determining the uncertainty or imprecision these errors induce in the solutions to the perturbation equations. The genuine deviations from FLRW will be those deviations which are larger than the error bars. These, too, can be traced in the same way through the perturbation equations to see whether or not they determine a model which is 'almost FLRW'.

Finally, it is important to recognize that there is a limit y_* to the null comoving radial coordinate out to which reliable data can be obtained [Stoeger et al. (1992a)]. The initial data on $C^-(p_0)$ for greater values of y than y_* will remain unknown. Consequently, we can really only perform our intergrations for $y \leq y_*$. Beyond that we really do not have the observational data needed to determine the geometry of space-time. We do, however, have CMWBR data from a redshift of $z \approx 1500$ ([Stoeger et al. (1995), Stoeger et al. (1997)]). However, that really does not serve to constrain adequately the space-time at more recent epochs and at less than those largest cosmological length scales. Note, however, that out to y_* we can, in this spherically symmetric case, both predict and retrodict to all allowed values of w ([Stoeger et al. (1992a)]). This is because of the assumed spherical symmetry, which effectively converts the usual hyperbolic equations of general relativity (for which prediction on the basis of given data is, strickly speaking, impossible) to ordinary differential equations for which these predictions to the future are legitimate.

6 Completing the Integration Procedure

In order to complete the integration of the perturbed field equations we introduce an improvement of the integration scheme recently developed by Araújo and Stoeger (1999) for the exact spherically symmetric case that simplifies our present task.

This modification of the integration scheme can be described schematically as follows:

(i) Having solved the null Raychaudhuri equation one finds $A(w_0, y)$ from equation (17).

(ii) Since y is chosen to be a comoving radial coordinate the functional

dependence of $A(w, y)$ with respect to y can not change as we move off our light cone.

(iii) We have mentioned above the freedom of rescaling the time coordinate w that is effected by choosing $A(w, 0)$. So, given (i) and (ii) above, this freedom effectively corresponds to choosing the functional dependence of $A(w, y)$ with respect to w in any way we like constrained only by the form of $A(w_0, y)$. In our expression for $A(w_0, y)$ is hidden an implicit dependence on w. We need to extract that dependence and make it explicit, so that we can then determine the general dependence of A on w and proceed with the integration. In general, this is not simply achieved by replacing w_0 with w because besides the w_0 dependence arising from setting $w = w_0$ when we write equation (8), there may be another part of the w_0 dependence which derives from integration constants of the null Raychaudhuri equation and remains through the entire problem. With the aim of clarifying these comments we take for instance the expression for $A(w_0, y)$ that is obtained from (17) and the solution to the null Raychaudhuri equation with FLRW $k = 0$ data ([Araújo and Stoeger (1999)]), that is,

$$A(w_0, y) = \frac{2}{H_0 w_0}\left(1 - \frac{y}{w_0}\right)^2 \tag{46}$$

where $H_0 \equiv H(w_0, y)$ is the Hubble parameter. At this point assume that we arbitrarily set the w dependence for A and proceed with the integration. The next step is then the solution of equations (28) and (30) to determine B and C respectively. Their general solutions are:

$$B = -\int A' dw + l(y) \tag{47}$$

where $l(y)$ is determined from the condition $A(w_0, y) = B(w_0, y)$, and

$$C = \left[\frac{3}{2}(-\omega_0)^{1/2}\int A dw + h(y)\right]^{2/3} \tag{48}$$

where $h(y)$ is determined from the data $r_0 = C(w_0, y)$.

Now, we note that examining the central conditions (15) on $C(w, y)$ force $B(w, y)$ to be of a certain form regarding its w dependence, which in turn, through equation (47) constrain $A(w, y)$ to have a definite form. In the present example it is clear that unless

$$A(w, y) = \frac{2}{H_0 w_0}\left(\frac{w - y}{w_0}\right)^2 \tag{49}$$

the central conditions and the form $A(w_0, y)$ given by the data and the solution of the null Raychaydhuri equation will not be satisfied.

(iv) $B(w, y)$ and $C(w, y)$ are then determined by integrating equations (28) and (30) with respect to w. The arbitrary functions of y that arise from these integrations are determined by the conditions $A(w_0, y) = B(w_0, y)$ and $C(w_0, y) = r_0(y)$ respectively. $B(w, y)$ and $C(w, y)$ are further constrained by the fact that they have to satisfy the central conditions (15). Now, it is clear from an examination of these equations that unless $A(w, y)$ has a very specific functional dependence on w the resulting solutions $B(w, y)$ and $C(w, y)$ will not satisfy the central conditions. That implies that, although we can find solutions to the field equations, it does not guarantee that the null surface on which we assume we have the data is a past light cone of our world line ([Ellis et al. (1985)]). So we conclude that (i), (ii) and the central conditions (15) remove the freedom of rescaling the time coordinate w and completely determine $A(w, y)$. Thus, all the coordinate freedom in y and w has been used up, and this, together with the determination of the best-fit FLRW model to the data serves to specify the gauge (see Section 4).

We now proceed to show how we can solve the perturbed spherially symmetric field equations following the procedure described above.

Combining equations (9) and (45), we obtain to first order:

$$R(w_0, y) - \frac{1}{2}\frac{Z^2(w_0, y)}{R(w_0, y)} = A_0 \left[\frac{1}{1 + z_F(y)} - \frac{z_1(y)}{[1 + z_F(y)]^2}\right]. \tag{50}$$

Equating orders separately yields

$$\frac{A_0}{1 + z_F(y)} = R(w_0, y) \Rightarrow A_0 = R_0 \tag{51}$$

and

$$Z^2(w_0, y) = 2A_0 R(w_0, y)\frac{z_1(y)}{[1 + z_F(y)]^2}. \tag{52}$$

In principle, the freedom of rescaling the time coordinate w together with our choice of a comoving radial coordinate y means that we can set the time dependency of Z^2 in any way we like constrained only by the form of $Z^2(w_0, y)$. So at this stage, given this choice, we already have $Z^2(w, y)$, which is fundamental for proceeding with the integration of the remaining field equations for the determination of $\beta^2(w, y)$ and $h_{22}(w, y)$. But, as we

discussed above, this freedom does not really exist because of the further constraints imposed by the central conditions. So in order to proceed with the integration we have to set the correct time dependency of Z^2 at this point.

Substituting (18) and (19) into (28) and collecting the first order terms yields

$$\left(\frac{\beta^2}{R}\right)^{\cdot} = \frac{1}{2}\left(\frac{Z^2}{R}\right)' - \frac{1}{2}\left(\frac{Z^2}{R}\right)^{\cdot}. \tag{53}$$

Integrating (53) with respect to w we find

$$\frac{\beta^2}{R} = \frac{1}{2}\int\left(\frac{Z^2}{R}\right)' dw - \frac{1}{2}\frac{Z^2}{R} + g(y), \tag{54}$$

where $g(y)$ is determined by the condition $\beta^2(w_0, y) = -Z^2(w_0, y)$.

In order to complete the solution, we have to find $h_{22}(w, y)$ from the perturbed form of equation (30). But first we must calculate the "potential" $W(y)$ and $\omega_0(y)$ from equations (29) and (32), respectively. That in turns involves the calculation of $\dot{h_{22}}(w_0, y)$. To do this we must solve the perturbed form of equation (27) on our past light cone $w = w_0$. Using equations (18), (19) and (20), the first order perturbation equation is given by

$$\begin{aligned}
&\frac{1}{2}\left[\left(\frac{h_{22}}{R\hat{f}}\right)^{\cdot}\right]' + \frac{(R\hat{f})'}{2R\hat{f}}\left(\frac{h_{22}}{R\hat{f}}\right)^{\cdot} = -\frac{(R\hat{f})^{\cdot}}{2R\hat{f}}\left(\frac{h_{22}}{R\hat{f}}\right)' \\
&+\frac{h_{22}}{2R^3\hat{f}^3}(R\hat{f})^{\cdot}(R\hat{f})' + \dot{R}\left[\frac{1}{2R}\left(\frac{h_{22}}{R\hat{f}}\right)' - \frac{Z^2}{2R^3}(R\hat{f})' - \frac{\beta^2}{R^3}(R\hat{f})'\right] \\
&+\frac{1}{2R}\left(\frac{Z^2}{R}\right)^{\cdot}(R\hat{f})' - \frac{R^2 h_{22}}{4R^3\hat{f}^3} - \frac{(R\hat{f})'}{2R\hat{f}}\left(\frac{h_{22}}{R\hat{f}}\right)' + \frac{h_{22}[(R\hat{f})']^2}{4R^3\hat{f}^3} \\
&+\frac{1}{R}\left(\frac{\beta^2}{R}\right)^{\cdot}(R\hat{f})' + \frac{\beta^2}{2R\hat{f}} + \frac{Z^2[(R\hat{f})']^2}{2R^3\hat{f}} + \frac{\beta^2[(R\hat{f})']^2}{2R^3\hat{f}}.
\end{aligned} \tag{55}$$

Since on our light cone $\beta^2(w_0, y) = -Z^2(w_0, y)$ we obtain at $w = w_0$

$$\left[\left(\frac{h_{22}}{R\hat{f}}\right)^{\cdot}\right]' + \frac{(R\hat{f})'}{R\hat{f}}\left(\frac{h_{22}}{R\hat{f}}\right)^{\cdot} = -\frac{(R\hat{f})^{\cdot}}{R\hat{f}}\left(\frac{h_{22}}{R\hat{f}}\right)'$$

$$
\begin{aligned}
&+\frac{h_{22}}{R^3\hat{f}^3}(R\hat{f})^{\cdot}(R\hat{f})' + \dot{R}\left[\frac{1}{R}\left(\frac{h_{22}}{R\hat{f}}\right)' + \frac{Z^2}{R^3}(R\hat{f})'\right] \\
&+\frac{1}{R}\left(\frac{Z^2}{R}\right)^{\cdot}(R\hat{f})' - \frac{R^2 h_{22}}{2R^3\hat{f}^3} - \frac{(R\hat{f})'}{R\hat{f}}\left(\frac{h_{22}}{R\hat{f}}\right)' + \frac{h_{22}[(R\hat{f})']^2}{2R^3\hat{f}^3} \\
&+\frac{2}{R}\left(\frac{\beta^2}{R}\right)^{\cdot}(R\hat{f})' - \frac{Z^2}{R\hat{f}}. \qquad (56)
\end{aligned}
$$

This is a standard linear equation whose integrating factor can be easily calculated and yields the solution

$$
\begin{aligned}
\left(\frac{h_{22}}{R\hat{f}}\right)^{\cdot}(w_0, y) \;=\; & \frac{1}{R\hat{f}}\int_0^y \left\{-(R\hat{f})^{\cdot}\left(\frac{h_{22}}{R\hat{f}}\right)' + \frac{h_{22}}{R^2\hat{f}^2}(R\hat{f})^{\cdot}(R\hat{f})'\right. \\
& +\dot{R}\left[\left(\frac{h_{22}}{R\hat{f}}\right)'\hat{f} + \frac{Z^2}{R^2}(R\hat{f})'\hat{f}\right] + \left(\frac{Z^2}{R}\right)^{\cdot}(R\hat{f})'\hat{f} \\
& -\frac{R^2 h_{22}}{2R^2\hat{f}^2} - (R\hat{f})'\left(\frac{h_{22}}{R\hat{f}}\right)' + \frac{h_{22}[(R\hat{f})']^2}{2R^2\hat{f}^2} \\
& \left. + 2\left(\frac{\beta^2}{R}\right)^{\cdot}(R\hat{f})'\hat{f} - Z^2\right\} d\tilde{y}. \qquad (57)
\end{aligned}
$$

We can now use these results to construct the potential $W(y)$ given in equation (29). Note that knowing $C^{\cdot}$, C', B and A at any one time, for instance $w = w_0$, determines $W(y)$ for all times since it is independent of w. In our case here we have just determined $h^{\cdot}_{22}(w_0, y)$. From the data we know $h'_{22}(w_0, y)$ and the from the solution to the null Raychaudhuri equation we have $Z^2(w_0, y) = -\beta^2(w_0, y)$. We can then write (evaluating at $w = w_0$)

$$
W(y) = W_F(y) + W_1(y) \qquad (58)
$$

where,

$$
\begin{aligned}
W_F(y) &= \frac{1}{R}[(R\hat{f})^{\cdot} + (R\hat{f})'] \\
W_1(y) &= \frac{Z^2}{2R^3}[(R\hat{f})^{\cdot} - (R\hat{f})'] - \frac{\beta^2}{R^3}(R\hat{f})' + \frac{1}{2R}\left[\left(\frac{h_{22}}{R\hat{f}}\right)^{\cdot} + \left(\frac{h_{22}}{R\hat{f}}\right)'\right] \qquad (59)
\end{aligned}
$$

At this stage, since we have already solved the null Raychaudhuri equation in Section 5 to find $z = z(y)$, we can write ω_0 given in (24) as a function of y to first order as

$$\omega_0(y) = \omega_F(y) + \omega_1(y) \tag{60}$$

where,

$$\begin{aligned}
\omega_{0_F} &= -\frac{R\hat{f}}{2} + \frac{1}{A_0}(R\hat{f})'(R\hat{f})W_F(1+z_F) - \frac{1}{2A_0^2}(R\hat{f})[(R\hat{f})']^2(1+z_F)^2 \\
\omega_{0_1} &= -\frac{h_{22}}{4R\hat{f}} + \frac{1}{A_0}(1+z_F)\left[(R\hat{f})'(R\hat{f})W_1 + \frac{1}{2}\left(\frac{h_{22}}{R\hat{f}}\right)'(R\hat{f})W_F\right. \\
&\quad \left. + \frac{h_{22}}{2R\hat{f}}(R\hat{f})'W_F\right] + \frac{1}{A_0}(R\hat{f})'(R\hat{f})W_F z_1 - \frac{1}{2A_0^2}(1+z_F)^2 \\
&\quad \times \left[\frac{1}{2}\frac{h_{22}}{R\hat{f}}[(R\hat{f})']^2 + (R\hat{f})'(R\hat{f})\left(\frac{h_{22}}{R\hat{f}}\right)'\right] - \frac{1}{A_0^2}(z_1 + z_F z_1) \\
&\quad \times \left[(R\hat{f})[(R\hat{f})']^2\right].
\end{aligned} \tag{61}$$

Substituting (20), (58) and (60) into (30) and equating orders separately we obtain the following first order equation for h_{22}

$$\left(\frac{h_{22}}{R\hat{f}}\right)^{\cdot} - \frac{\omega_{0_F}}{R\hat{f}^2}F^{-1}(y)\frac{h_{22}}{R\hat{f}} = R\left(2W_F W_1 - 2\omega_{0_1}/R\hat{f}\right)F^{-1}(y) - \frac{Z^2}{R}F(y) \tag{62}$$

where $F \equiv \left(W_F^2 - 1 - 2\omega_{0_F}/R\hat{f}\right)^{1/2}$. Equation (62) is a standard linear equation whose general solution is

$$\begin{aligned}
\left(\frac{h_{22}}{R\hat{f}}\right)(w,y) &= \frac{1}{e^{a(y)w}}\left\{\int e^{a(y)w}b(y)dw + c(y)\right\} \\
&= \frac{1}{e^{a(y)w}}\left\{b(y)\left[\frac{e^{a(y)w}}{a(y)}\right] + c(y)\right\} \\
&= \frac{b(y)}{a(y)} + (e^{a(y)w})^{-1}c(y)
\end{aligned} \tag{63}$$

where,

$$
\begin{aligned}
a(y) &\equiv -\frac{\omega_{0_F}}{R\hat{f}^2}F^{-1}(y) \\
b(y) &\equiv R\left(2W_F W_1 - 2\omega_{0_1}/R\hat{f}\right)F^{-1}(y) - \frac{Z^2}{R}F(y)
\end{aligned}
\tag{64}
$$

and $c(y)$ is a function determined by the data $h_{22}(w_0, y)$.

To summarize, once we solve the null Raychaudhuri equation we find $Z^2(w_0, y)$ from equation (52), then according to our discussion in the begining of this section concerning the use of the central conditions we determine $Z^2(w, y)$. We then proceed to find $\beta(w, y)$ (equation (54)). Finally $h_{22}(w, y)$ (equation (63)) is determined completing the solution.

This procedure gives us the full solution to the first-order perturbed equations in terms of data functions for number counts and observer area distance on our past light cone. As we have already mentioned briefly in Section 5, we could go on in a similar way to solve the perturbation equations for second order and higher. In order to determine whether or not our perturbation treatment is valid, we shall have to look at higher-order solutions. Does the perturbation series we obtain converge, or at least give some evidence of converging, for the data functions $r_{0_+}(z)$ and $M_{0_+}(z)$ we input? If it does, and if those solutions remain convergent for all values of w, then we can affirm in a rigorous sense that our universe is 'almost FLRW' on the scales represented by the data. If they are not, then the perturbation treatment is invalid, and our universe is *not* 'almost FLRW' on those scales.

Of course, in doing this, we must also track separately the 'error component' and the 'genuine perturbation' component of the metric deviations from FLRW. We briefly indicated at the end of Section 5 how this can be done.

In the next paper, we shall present the solution to the general FLRW perturbation problem for observational data on our past light cone. That case will be similar in many ways, but much more complicated in others. In fact, it is only with the insights gained from this detailed study of the solution to the spherically symmetric case that we see how to proceed with the more general case.

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